

On a Polynomial Solutions of a Diophantine Equation

Manju Somanath*

K. Raja**

J. Kannan***

Abstract

Let $P := P(t)$ be a polynomial in $\mathbf{Z}[x]$. In this paper, we consider the polynomial solutions of Diophantine equation $D: K^2 - 56R^2 - 32K - 224R - 224 = 0$. We also obtain some formulae and recurrence relations on the polynomial solution (K_n, R_n) of D .

Keywords:

Pell equation,
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Polynomial solution,
Continued fraction expansion.

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Author correspondence:

K. Raja,
Assistant Professor,
Department of Mathematics, National College, Bharathidasan University, Trichy, India.
Email: rajakonline@gmail.com

1. Introduction

A Diophantine equation is a polynomial equation $P(x_1, x_2, \dots, x_n) = 0$ where the polynomial P has integral coefficients and one is interested in solutions for which all the unknowns take integer values. For example, $x^2 + y^2 = z^2$ and $x = 3, y = 4, z = 5$ is one of its infinitely many solutions. Another example is $x + y = 1$ and all its solutions are given by $x = t, y = 1 - t$ where t passes through all integers. A third example is $x^2 + 4y = 3$. This Diophantine equation has no solutions, although note that $x = 0, y = \frac{3}{4}$ is a solution with rational values for the unknowns. Diophantine equations are rich in variety. Two – variable Diophantine equation have been a subject to extensive research, and their theory constitutes one of the most beautiful, most elaborate part of mathematics, which nevertheless still keeps some of its secrets for the next generation of researchers.

In this paper, we investigate positive integral solutions of the Diophantine equation $K^2 - 56R^2 - 32K - 224R - 224 = 0$ which is transformed into a Pell's equation and is solved by various methods.

2. THE DIOPHANTINE EQUATION $K^2 - 56R^2 - 32K - 224R - 224 = 0$

Consider the Diophantine equation

$$D: K^2 - 56R^2 - 32K - 224R - 224 = 0 \quad (1)$$

to be solved over \mathbf{Z} . It is not easy to solve and find the nature and properties of the solutions of (1). So we apply a linear transformation T to (1) to transfer to a simpler form for which we can determine the integral solutions.

$$\text{Let } T: \begin{cases} K = x + h \\ R = y + k \end{cases} \quad (2)$$

be the transformation where $h, k \in \mathbf{Z}$.

*Assistant Professor, Department of Mathematics, National College, Trichy, Tamil Nadu, India.

**Assistant Professor, Department of Mathematics, National College, Trichy, Tamil Nadu, India.

***Research Scholar, Department of Mathematics, National College, Trichy, Tamil Nadu, India.

Applying T to D , we get

$$T(D) = \tilde{D}: (x+h)^2 - 56(y+k)^2 - 32(x+h) - 224(y+k) = 224 \quad (3)$$

Equating the coefficients of x and y to zero, we get $h = 16$ and $k = -2$. Hence for $K = x + 16$ and $R = y - 2$, we have the Diophantine equation

$$\tilde{D}: x^2 - 56y^2 = 256 \quad (4)$$

which is a Pell equation. Now we try to find all integer solutions (x_n, y_n) of \tilde{D} and then we can retransfer all results from \tilde{D} to D by using the inverse of T .

Theorem 2.1:

Let \tilde{D} be the Diophantine equation in (4). Then

- (i) The continued fraction expansion of $\sqrt{56}$ is

$$\sqrt{56} = [7; \overline{2, 14}]$$

- (ii) The fundamental solution of $x^2 - 56y^2 = 1$ is $(u_1, v_1) = (15, 2)$

- (iii) For $n \geq 4$,

$$u_n = 31(u_{n-1} - u_{n-2}) + u_{n-3}$$

$$v_n = 31(v_{n-1} - v_{n-2}) + v_{n-3}$$

Proof:

- (i) The continued fraction expansion of $\sqrt{56} = 7 + (\sqrt{56} - 7)$

$$\begin{aligned} &= 7 + \frac{1}{\frac{1}{\sqrt{56}-7}} \\ &= 7 + \frac{1}{\frac{\sqrt{56}+7}{7}} \\ &= 7 + \frac{1}{2 + \frac{\sqrt{56}-7}{7}} \\ &= 7 + \frac{1}{2 + \frac{1}{\frac{7}{\sqrt{56}-7}}} \\ &= 7 + \frac{1}{2 + \frac{1}{\frac{1}{\sqrt{56}+7}}} \\ &= 7 + \frac{1}{2 + \frac{1}{14 + (\sqrt{56}-7)}} \end{aligned}$$

Therefore the continued fraction expansion of $\sqrt{56}$ is

$$[7; \overline{2, 14}]$$

- (ii) It is easily seen that

$$(u_1, v_1) = (15, 2)$$

is a solution of $x^2 - 56y^2 = 1$ since

$$\begin{aligned} x_1^2 - 56y_1^2 &= (15)^2 - 56(2)^2 \\ &= 1 \end{aligned}$$

- (iii) Note that by (3), if $(u_1, v_1) = (15, 2)$ is the fundamental solution of $x^2 - 56y^2 = 1$, then the other solutions (u_n, v_n) of $x^2 - 56y^2 = 1$ can be derived by using the equalities

$$(u_n + v_n\sqrt{56}) = (u_1 + \sqrt{56}v_1)^n \text{ for } n \geq 2, \text{ in other words,}$$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & 56v_1 \\ 2 & u_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $n \geq 2$. Therefore it can be shown by induction on n that

$$u_n = 31(u_{n-1} - u_{n-2}) + u_{n-3}$$

and also

$$v_n = 31(v_{n-1} - v_{n-2}) + v_{n-3}, \quad \text{for } n \geq 4.$$

Now we consider the problem

$$x^2 - 56y^2 = 256$$

Note that we denote the integer solutions of $x^2 - 56y^2 = 256$ by (x_n, y_n) , and denote the integer solutions of $x^2 - 56y^2 = 1$ by (u_n, v_n) . Then we have the following theorem.

Theorem 2.2:

Define a sequence $\{(x_n, y_n)\}$ of positive integers by

$$(x_1, y_1) = (240, 32)$$

and

$$x_n = 240u_{n-1} + 1792v_{n-1}$$

$$y_n = 32u_{n-1} + 240v_{n-1},$$

where $\{(u_n, v_n)\}$ is a sequence of positive solutions of $x^2 - 56y^2 = 1$. Then

(1) (x_n, y_n) is a solution of $x^2 - 56y^2 = 256$ for any integer $n \geq 1$.

(2) For $n \geq 2$,

$$x_{n+1} = 15x_n + 112y_n$$

$$y_{n+1} = 2x_n + 15y_n.$$

(3) For $n \geq 4$

$$x_n = 31(x_{n-1} - x_{n-2}) + x_{n-3}$$

$$y_n = 31(y_{n-1} - y_{n-2}) + y_{n-3}$$

Proof:

(1) It is easily seen that

$$(x_1, y_1) = (240, 32)$$

is a solution of $x^2 - 56y^2 = 256$ since

$$\begin{aligned} x_1^2 - 56y_1^2 &= (240)^2 - 56(32)^2 \\ &= 256(15^2 - 56(2^2)) \\ &= 16(1) \\ &= 16 \end{aligned}$$

Note that by definition, (u_{n-1}, v_{n-1}) is a solution of $x^2 - 56y^2 = 1$, that is,

$$u_{n-1}^2 - 56v_{n-1}^2 = 1. \quad (7)$$

Also we see as above that (x_1, y_1) is a solution of $x^2 - 56y^2 = 256$, that is,

$$x_1^2 - 56y_1^2 = 256. \quad (8)$$

Applying (7) and (8), we get

$$\begin{aligned} x_n^2 - 56y_n^2 &= (240u_{n-1} + 1792v_{n-1})^2 - 56(32u_{n-1} + 240v_{n-1})^2 \\ &= u_{n-1}^2(2^8) - v_{n-1}^2(2^8(56)) \\ &= 2^8(u_{n-1}^2 - 56v_{n-1}^2) \\ &= 2^8 \end{aligned}$$

Therefore (x_n, y_n) is a solution of $x^2 - 56y^2 = 2^8$.

(2) Recall that

$$x_{n+1} + y_{n+1}\sqrt{d} = (u_1 + v_1\sqrt{d})(x_n + y_n\sqrt{d})$$

Therefore

$$x_{n+1} = u_1x_n + v_1y_nd \quad \text{and} \quad y_{n+1} = v_1x_n + u_1y_n$$

So

$$x_{n+1} = 15x_n + 112y_n \quad \text{and} \quad y_{n+1} = 2x_n + 15y_n \quad (*)$$

Since $u_1 = 15$ and $v_1 = 2$.

(3) Applying the equalities

$$x_n = 2^3(13)u_{n-1} + 2^4(42)v_{n-1} \quad \text{and} \quad x_{n+1} = 13x_n + 84y_n$$

We find by induction on n that

$$x_n = 31(x_{n-1} - x_{n-2}) + x_{n-3}$$

for $n \geq 4$. Similarly it can be shown that

$$y_n = 31(y_{n-1} - y_{n-2}) + y_{n-3}.$$

Corollary 2.3:

The base of the transformation T in (2) is the fundamental solution of \tilde{D} , that is $T[h; k] = \{h, k\} = \{x_1, y_1\}$.

Proof: We proved that $(x_1, y_1) = (240, 32)$ is the fundamental solution of \tilde{D} . Also we showed that $h = 16$ and $k = -2$. So the base of T is $T[h, k] = \{16, -2\}$ as we claimed. We saw as above that the Diophantine equation D could be transformed into the Diophantine equation \tilde{D} via the transformation T . Also we showed that $K = x + 16$ and $R = y - 2$. So we can retransfer all results from \tilde{D} to D by using the inverse of T . Thus we can give the following main theorem

Theorem 2.4:

Let D be the Diophantine equation in (1), Then

- (1) The fundamental solution of D is $(K_1, R_1) = (256, 30)$.
- (2) Define the sequence $\{(K_n, R_n)\}_{n \geq 1} = \{(x_n + 16, y_n - 2)\}$, where $\{(x_n, y_n)\}$ defined in (*). Then (K_n, R_n) is a solution of D . So it has infinitely many solutions $(K_n, R_n) \in \mathbf{Z} \times \mathbf{Z}$.
- (3) The solution (K_n, R_n) satisfy

$$K_n = 15K_{n-1} + 112R_{n-1}$$

$$R_n = 2K_{n-1} + 15R_{n-1} - 4$$

- (4) The solutions (K_n, R_n) satisfy the recurrence relations

$$K_n = 31(K_{n-1} - K_{n-2}) + K_{n-3}$$

$$R_n = 31(R_{n-1} - R_{n-2}) + R_{n-3}$$

Proof:

- (1) It is easily seen that $(K_1, R_1) = (256, 30)$ is the fundamental solution of D since $256^2 - 56(30)^2 - 32(256) - 224(30) - 224 = 0$.
- (2) We prove it by induction. Let $n = 1$. Then $(K_1, R_1) = (x_1 + 16, y_1 - 2) = (256, 30)$ which is the fundamental solution and so is a solution of D . Let us assume that the Diophantine equation in (1) is satisfied for $n - 1$, that is, $(x_{n-1} + 16)^2 - 56(y_{n-1} - 2)^2 - 32(x_{n-1} + 16) - 224(y_{n-1} - 2) - 224 = 0$. We want to show that this equation is also satisfied for n .

$$\begin{aligned} K^2 - 56R^2 - 32K - 224R - 224 &= (x_n + 16)^2 - 56(y_n - 2)^2 - 32(x_n + 16) - 224(y_n - 2) - 224 \\ &= x_n^2 - 56y_n^2 - 256 \\ &= 0 \end{aligned} \quad (x_n \text{ and } y_n \text{ solutions of } \tilde{D}).$$

So $(K_n, R_n) = (x_n + 16, y_n - 2)$ is also a solution D .

- (3) From (*) $x_n = 15x_{n-1} + 112y_{n-1}$.

Adding 16 on both sides,

$$x_n + 16 = 15x_{n-1} + 112y_{n-1} + 16$$

We know that $K_n = x_n + 16$ and $R_n = y_n - 2$

Therefore, $x_n = K_n - 16$ and $y_n = R_n + 2$

$$\begin{aligned} x_n + 16 &= 13x_{n-1} + 84y_{n-1} + 16 \\ (K_n - 16) + 16 &= 15(K_{n-1} - 16) + 112(R_{n-1} + 2) + 16 \end{aligned}$$

We get, $K_n = 15K_{n-1} + 112R_{n-1}$ (9)

Similarly, $R_n = 2K_{n-1} + 15R_{n-1} - 4$ (10)

- (4) We prove that x_n satisfy the recurrence relation. For $n = 4$, we get $K_1 = 256, K_2 = 7200,$

$K_3 = 215296, K_4 = 6451232$. Hence

$$\begin{aligned} K_4 &= 31(K_3 - K_2) + K_1 \\ &= 31(215296 - 7200) + 256 \end{aligned}$$

So $K_4 = 31(K_3 - K_2) + K_1$ is satisfied for $n = 4$. Let us assume that this relation is satisfied for $n - 1$, that is,

$$K_{n-1} = 31(K_{n-2} - K_{n-3}) + K_{n-4} \quad (11)$$

Then applying the previous assertion, (9) and (11), we conclude that $K_n = 31(K_{n-1} - K_{n-2}) + K_{n-3}$ for $n \geq 4$.

Now prove that y_n satisfy the recurrence relation. For $n = 4$, we get $R_1 = 30, R_2 = 958, R_3 = 28766, R_4 = 862078$. Hence

$$\begin{aligned} R_4 &= 31(R_3 - R_2) + R_1 \\ &= 31(28766 - 958) + 30 \end{aligned}$$

So $R_n = 31(R_{n-1} + R_{n-2}) - R_{n-3}$ is satisfied for $n = 4$. Let us assume that this relation is satisfied for $n - 1$, that is,

$$R_{n-1} = 31(R_{n-2} - R_{n-3}) + R_{n-4} \quad (12)$$

Then applying the previous assertion, (10) and (12), we conclude that $R_n = 31(R_{n-1} - R_{n-2}) + R_{n-3}$, for $n \geq 4$.

3. Conclusion

Diophantine equations are rich in variety. There is no universal method for finding all possible solutions (if it exists) for Diophantine equations. The method looks to be simple but it is very difficult for reaching the solutions.

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